数理解析研究所講究録
799巻 1992年 107-112

ON SYMMETRIC ALGEBRAS

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Throughout this note, all algebras and modules are finite dimensional over an algebraically closed field K.

An algebra Λ is said to be symmetric if the regular module Λ is isomorphic to its dual $D(\Lambda)$ as a bi-Λ-module, where $D = \mathrm{Hom}(-,K)$. It is well known that any block ideal of KG, the group algebra of a finite group G, is symmetric. Therefore, it seems interesting to know the way of constructing symmetric algebras from ring theoretical view point.

The trivial extension algebra $A\ltimes D(A)$, for any algebra A, is symmetric. This is also well known and the trivial extension algebras $A\ltimes D(A)$ have been used for studyng representation-finite self-injective algebras by many authors. Our costruction is given by generalizing trivial extension algebras.

We start by giving the definition of nilpotent Morita context. We call a linear map $\phi: {}_AN\otimes_A N_A \to {}_AN_A$ a (generalized) Morita context if ϕ is associative, i.e., $\phi(\phi(n\otimes n')\otimes n'') = \phi(n\otimes\phi(n'\otimes n''))$. It is easy to see that $B = A \oplus N$ has an algebra-structure by $(a,n)\cdot(a',n') = (a\cdot a', a\cdot n' + n\cdot a' + \phi(n\otimes n'))$. If $\phi = 0$ then B is the same with $A \ltimes N$. In the algebra B, N is an ideal. We call ϕ nilpotent if the ideal N is nilpotent. In this case we have $J(B) = J(A) \oplus N$.

In the case ${}_AN_A = {}_AM_A \oplus {}_AS_A$ and ϕ is given by $\phi((m,s)\otimes(m',s')) = (\varphi(m\otimes m'), \psi(m\otimes m'))$, where $\varphi: {}_AM\otimes_A M_A \to {}_AM_A$ and $\psi: {}_AM\otimes_A M_A \to {}_AS_A$, we dente $\phi = (\varphi,\psi)$. Then we know that (φ,ψ) is a (nilpotent) Morita context if and only if 1) φ is a (nilpotent) Morita context and 2) $\psi(\varphi(m\otimes m')\otimes m'') = \psi(m\otimes\varphi(m'\otimes m''))$.

In the case (φ,ψ) is a nilpotent Morita context, we denote the algebra $A\oplus M\oplus S$ with the multiplication $(a,m,s)\cdot(a',m',s')$ $= (a\cdot a', a\cdot m' + m\cdot a' + \varphi(m\otimes m'), a\cdot s' + s\cdot a' + \psi(m\otimes m'))$ by $\Lambda(\varphi,\psi)$.

Now, by using nilpotent Morita contexts, we define QF-systems. We call (φ,χ,θ,f) a QF-system if 1) A is an algebra and ${}_AM_A$ is a bimodule, 2) $\varphi: {}_AM\otimes_A M_A \to {}_AM_A$ is a nilpotent Morita context, 3) $\chi: M_A \to D(M)_A$ is an isomorphism with the property $\chi(m)(\varphi(m'\otimes m''))$ $= \chi(\varphi(m\otimes m'))(m'')$ and 4) θ is an algebra automorphism of A and $f \in D(\mathrm{Im}\varphi)$ such that $(\chi(am) - \theta(a)\chi(m))(m') = (fa - \theta(a)f)(\varphi(m\otimes m'))$.

Let us denote by ${}_\theta D(A)_A$ the bimodule $D(A)$ defined as the following manner: $(a\cdot q)(a') = q(a'\cdot\theta(a))$, $(q\cdot a)(a') = q(a\cdot a')$ for $a, a' \in A$ and $q \in D(A)$. Then, by defining $\psi(m\otimes m')(a) = \chi(m)(m'\cdot a) - f(\varphi(m\otimes m'\cdot a))$, we get a nilpotent Morita context (φ,ψ) on the bimodule ${}_AM_A \oplus {}_\theta D(A)_A$. We will denote the algebra $\Lambda(\varphi,\psi)$ by $\Lambda(\varphi,\chi,\theta,f)$.

<u>Theorem 1</u>. For any QF-system (φ,χ,θ,f), the algebra $\Lambda(\varphi,\chi,\theta,f)$ is Frobenius, i.e., $\Lambda(\varphi,\chi,\theta,f)$ is isomorphic to its dual as a one-sided module.

<u>Theorem 2</u>. Assume Λ is basic, indecomposable and self-injective. If Λ is not isomorphic to K, there exists a QF-system (φ,χ,θ,f) and $\Lambda \cong \Lambda(\varphi,\chi,\theta,f)$.

We call a QF-system $(\varphi,\chi,id_A,0)$ a symmetric QF-system (or SQF-system for short) if χ is symmetric, i.e., $\chi(m)(m') = \chi(m')(m)$. Precisely describing, (φ,χ) is an SQF-system if 1) A is an algebra and ${}_AM_A$ is a bimodule, 2) φ is a nilpotent Morita context defined on ${}_AM_A$ and 3) $\chi: {}_AM_A \to {}_AD(M)_A$ is an isomorphism with the properties $\chi(m)(m') = \chi(m')(m)$ and $\chi(\varphi(m\otimes m'))(m'') = \chi(m)(\varphi(m'\otimes m''))$.

Corresponding to the above results, we have

<u>Theorem 3</u>. For any SQF-system (φ,χ), the algebra $\Lambda(\varphi,\chi)$ is symmetric.

Theorem 4. Assume Λ is basic, indecomposable and symmetric. If Λ is not isomorphic to K, there exists an SQF-system (φ,χ) and $\Lambda \cong \Lambda(\varphi,\chi)$.

Now, let (φ,χ) be an SQF-system and P_A a progenerator with $B = \mathrm{End}(P_A)$. Then, it is easy to see that, on the modules ${}_BM^*{}_B = {}_BP\otimes_A M\otimes_A P^*{}_B$ and ${}_BD(B)_B = {}_BP\otimes_A D(A)\otimes_A P^*{}_B$, where ${}_AP^*{}_B = \mathrm{Hom}(P_A, A_A)$, we have an SQF-system (φ^*,χ^*) defined by

$$\varphi^*(p\otimes m\otimes h\otimes p'\otimes m'\otimes h') = p\otimes\varphi(m\otimes h(p')\cdot m')\otimes h'$$

$$\chi^*(p\otimes m\otimes h)(p'\otimes m'\otimes h') = \chi(m)(h(p')\cdot m'\cdot h'(p))$$

for $m, m' \in M$, $p, p' \in P$ and $h, h' \in P^*$. Further, it is checked that $\Lambda(\varphi^*,\chi^*) \cong \mathrm{End}(P\otimes_A\Lambda(\varphi,\chi)_{\Lambda(\varphi,\chi)})$. Therefore, we have

Corollary 5. For any symmetric algebra Λ, there exists an SQF-system (φ,χ) and $\Lambda \cong \Lambda(\varphi,\chi) \times S$, where S is a product of full matrix algebras (= the semi-simple part of Λ).

By the above corollary, we know that we have to study the way of constructing SQF-systems, in order to get symmetric algebras.

Here, we list some constructions of SQF-systems:

(Construction I) Let (φ_i,χ_i) be SQF-systems. Then, the direct sum $\bigoplus_i (\varphi_i,\chi_i)$ is again an SQF-system.

(Construction II) Let $\phi\colon {}_AI\otimes_A I_A \to {}_AI_A$ be a nilpotent Morita context. Then, by putting ${}_AM_A = {}_AI_A \oplus {}_AD(I)_A$ and

$$\varphi((x,q)\otimes(x',q')) = (\phi(x\otimes x'), q'(\phi(-\otimes x)) + q(\phi(x'\otimes -)))$$

$$\chi((x,q))((x',q')) = q'(x) + q(x') \quad \text{for} \quad x, x' \in I \quad \text{and} \quad q, q' \in D(I)$$

we have an SQF-system (φ,χ). We call this system the trivial extension of ϕ and denote it by $\phi \ltimes D(\phi)$. If ${}_AI_A$ is a nilpotent ideal of A and ϕ is given by $\phi(x\otimes y) = x\cdot y$, the multiplication in A, we denote more simply by $I \ltimes D(I)$.

(Construction III) Let (φ_0,χ_0) be an SQF-system defined on a bimodule ${}_AX_A$ and G a finite group. We put ${}_AM_A = \bigoplus_{g\in G} X^{(g)}$, $\varphi = \varphi_0 : {}_AX^{(g)}\otimes_A X^{(h)}{}_A \to {}_AX^{(g\cdot h)}{}_A$ and $\chi = \chi_0 :$ ${}_AX^{(g)}{}_A \to {}_AD(X^{(g^{-1})})_A$. It is easy to see that (φ,χ) is an SQF-system defined on ${}_AM_A$.

(Construction IV) Let E be a symmetric algebra with an isomorphism $d: {}_EE_E \to {}_ED(E)_E$. Assume there is an algebra map $\zeta: A \to E$. Then, putting ${}_AM_A = \bigoplus_{i=1}^{k-1} E^{(i)}$, $\varphi: {}_AE^{(i)}\otimes_A E^{(j)}{}_A \twoheadrightarrow {}_AE^{(i)}\otimes_E E^{(j)}{}_A \simeq {}_AE^{(i+j)}{}_A$ for $i+j \leqslant k-1$ and $\chi = d : {}_AE^{(i)}{}_A \to$ ${}_AD(E^{(k-i)})_A$, we have an SQF-system (φ,χ). We denote the algebra $\Lambda(\varphi,\chi)$ by $\Lambda_k(A,\zeta)$. For any module V_A, we may take $\mathrm{End}({}_KV)$ (= a full matrix algebra, and, therefore, symmetric) as E and ζ_V = the representation of V_A as ζ. In this case, the corresponding algebra will be denoted by $\Lambda_k(A,V_A)$.

(Construction V) Let (φ_X,χ_X) be an SQF-system defined on a bimodule ${}_A X_A$. Let $B = A \oplus X$ be the algebra defined by the multiplication $(a,x)\cdot(a',x') = (a\cdot a',\ a\cdot x' + x\cdot a' + \varphi_X(x\otimes x'))$.

Assume (φ_Y, χ_Y) is an SQF-system defined on a bimodule ${}_B Y_B$. Then, on the bimodule ${}_A X_A \oplus {}_A Y_A$, we can define an SQF-system (φ, χ) as follows:

$$\varphi((x,y)\otimes(x',y')) = (\varphi_X(x\otimes x') + \chi_X^{-1}(\chi_Y(y)(y'-)), x\cdot y' + y\cdot x' + \varphi_Y(y\otimes y')),$$

$$\chi((x,y))((x',y')) = \chi_X(x)(x') + \chi_Y(y)(y').$$